FFT

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COSC 581 Algorithm

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Research Interest: Machine Learning and Cybersecurity

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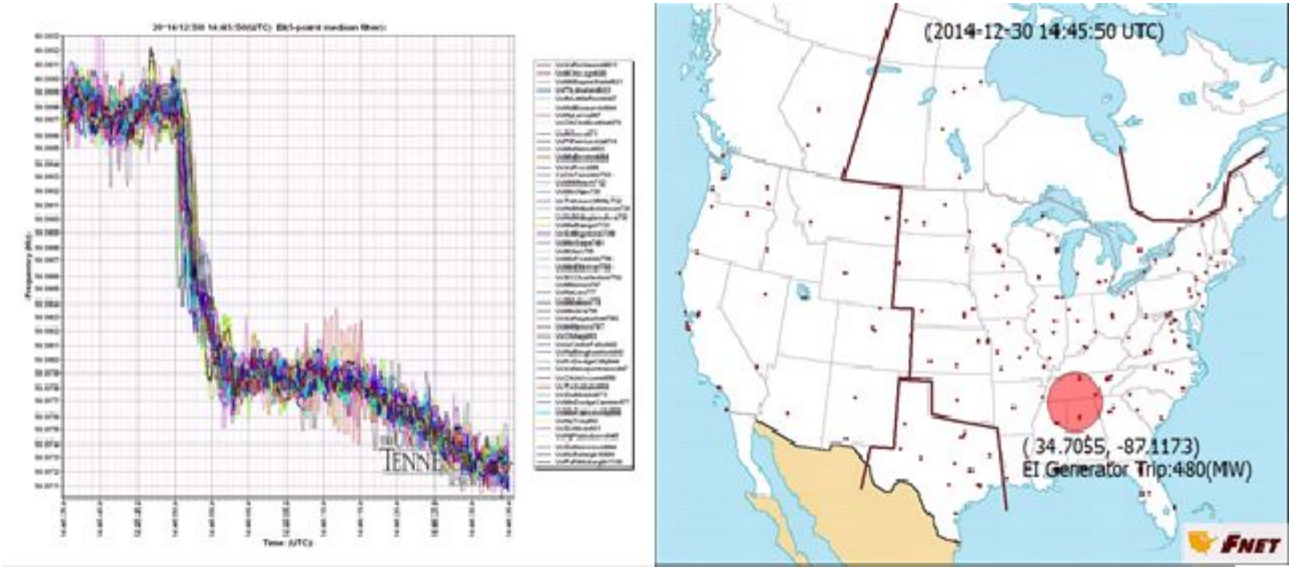
Research Interest:

Frequency Monitoring Network data analysis



Hobby: Japanese Anime, Basketball

Questions

1. Does the Fourier series prove that a non-periodic function is a sum of trigonometric functions?

2. What are the steps of divide & conquer approach?

3. Based on Euler's formula, a general point on a complex plane can be represented in what format?

Content

1. History
2. Fourier series
3. Fourier transform
4. Discrete Fourier transform (DFT)
5. Fast Fourier transform (FFT)
6. Some Applications
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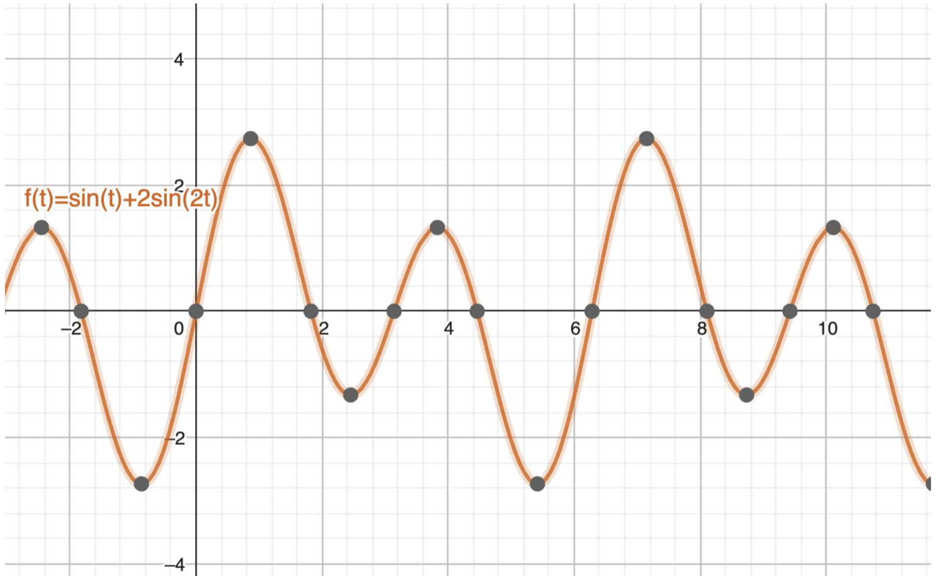
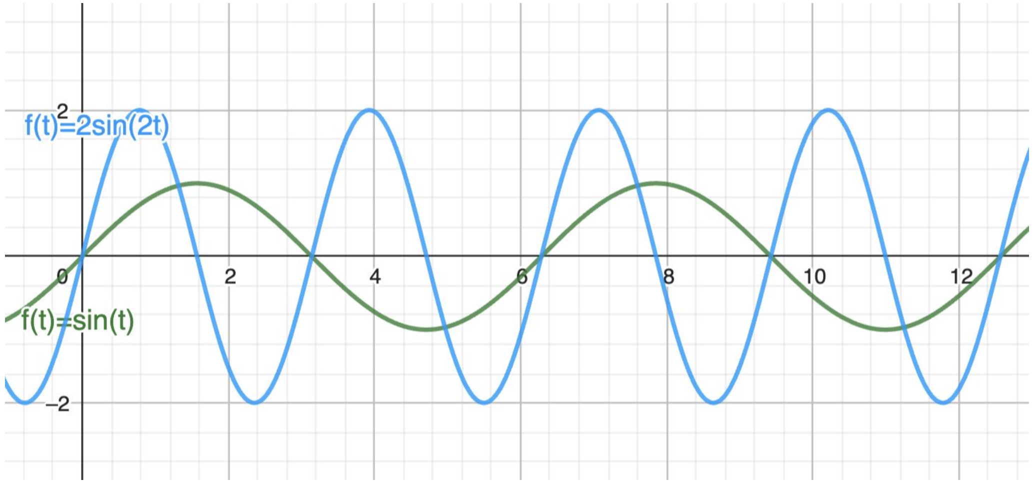
1. History

- 1807, Jean-Baptiste Joseph **Fourier** proposed **Fourier Series.** A **Fourier series** is an expansion of a periodic function into a sum of trigonometric functions [1]
- The **Fourier transform** (**FT**) is a transform that converts a function into a form that describes the frequencies present in the original function [2]
- The **discrete Fourier transform** (**DFT**) converts a finite sequence of equally-spaced samples of a function into a same-length sequence of equally-spaced samples of the discrete-time Fourier transform (DTFT), which is a complex-valued function of frequency
- 1965, J. W. **Cooley** and John **Tukey**. A **fast Fourier transform** (**FFT**) is an algorithm that computes the discrete Fourier transform (DFT) of a sequence, or its inverse (IDFT)[4]

[1] https://en.wikipedia.org/wiki/Fourier_series
[2] https://en.wikipedia.org/wiki/Fourier_transform
[3] https://en.wikipedia.org/wiki/Discrete_Fourier_transfo
[4] https://en.wikipedia.org/wiki/Fast_Fourier_transform

2. Fourier series

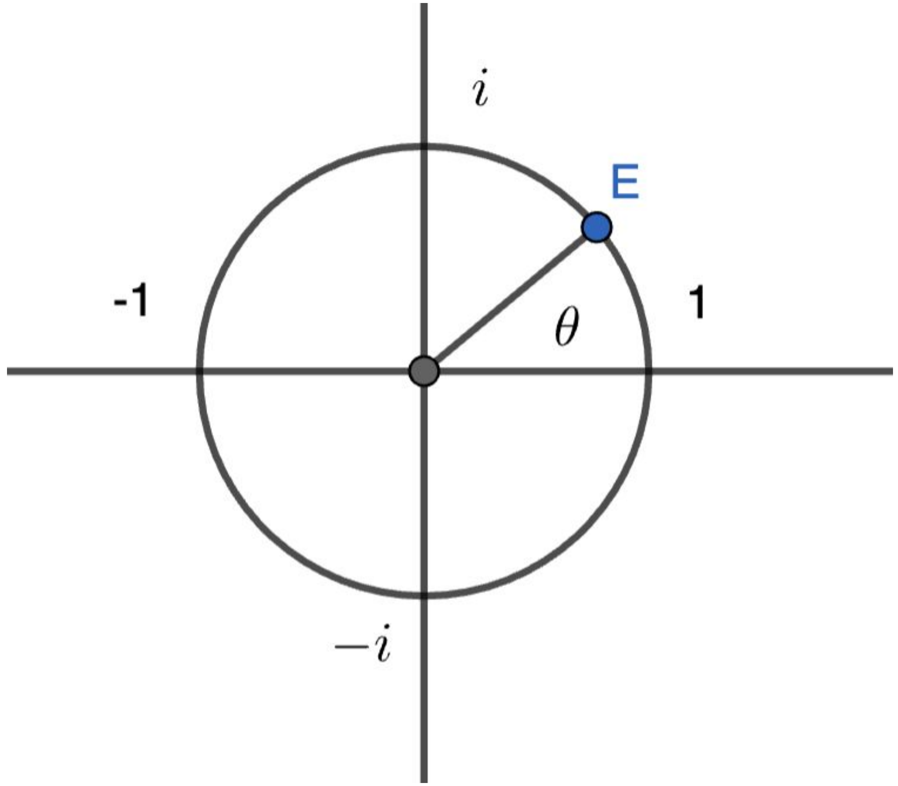




$$f(t) = \frac{a_0}{2} + \sum a_n sin(n\omega t + \varphi_n)$$

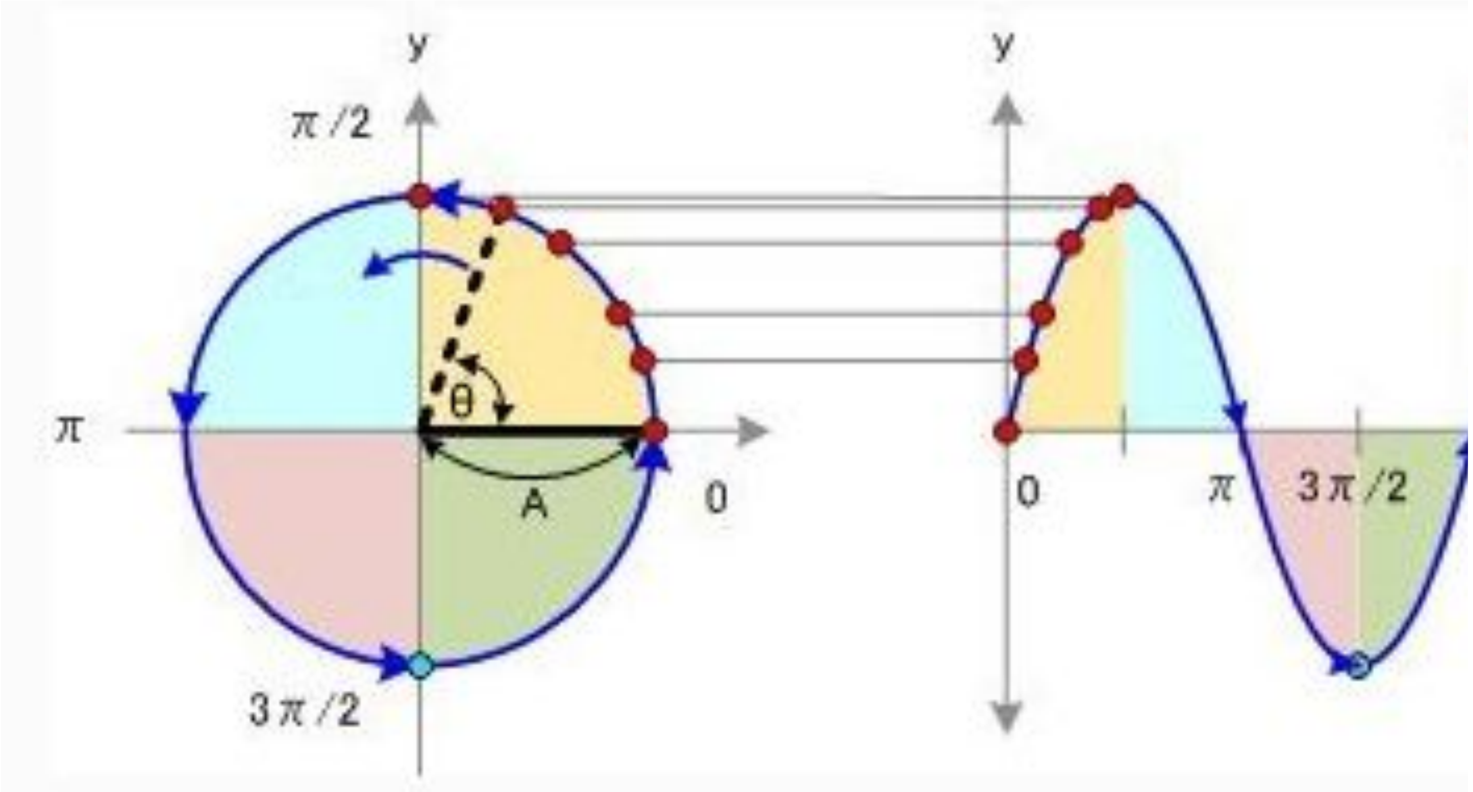
$$= \frac{a_0}{2} + \sum a_n sin(n\omega t) + \sum b_n cos(n\omega t)$$

3. Fourier Transform

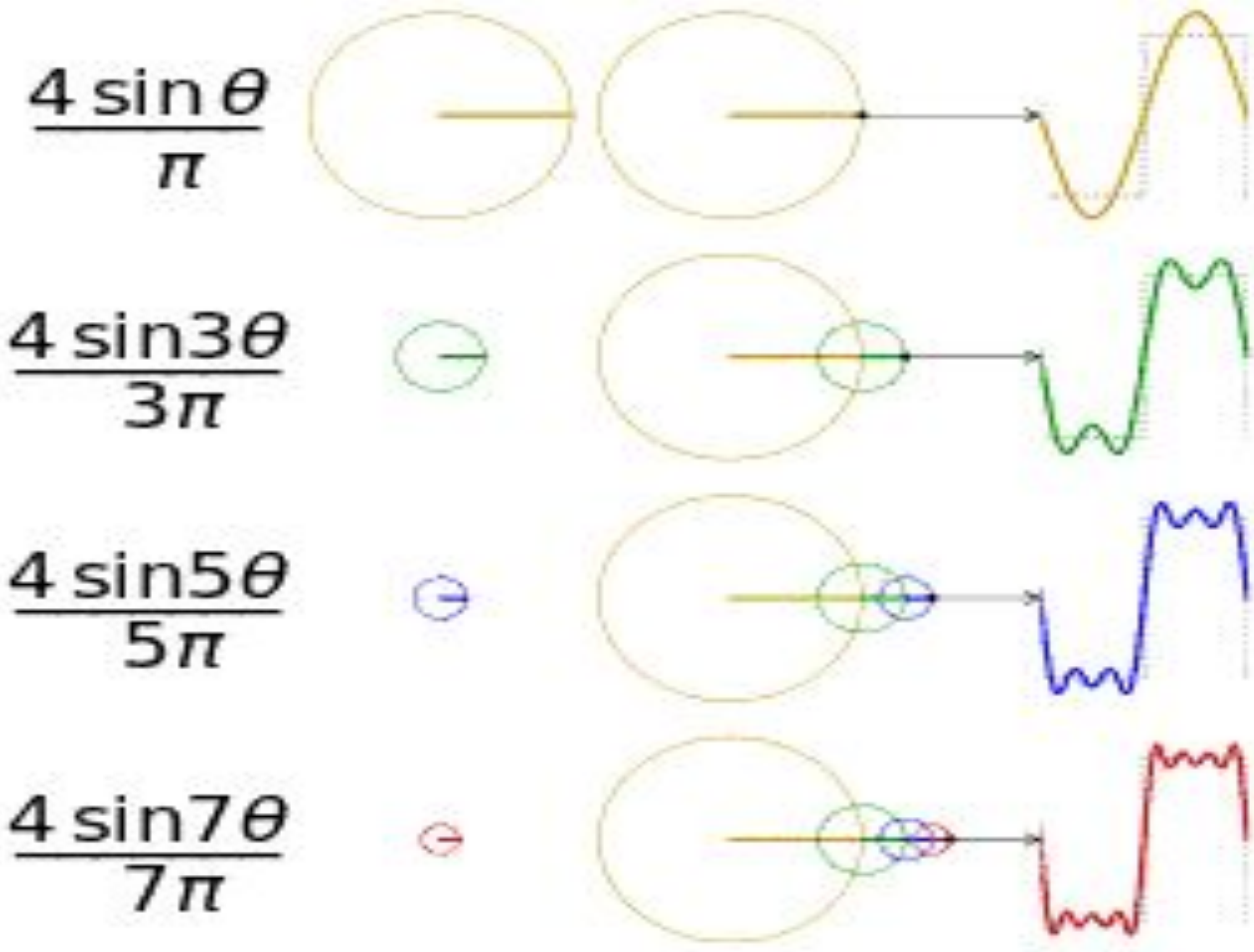


$$cos\theta + isin\theta = e^{i\theta}$$

$$\theta = \omega t \rightarrow e^{i\omega t}, e^{-i\omega t}$$



https://eipi10.cn/mathematics/2020/04/19/fourier_transform_2/



https://eipi10.cn/mathematics/2020/04/19/fourier_transform_2/

$$cos\theta + isin\theta = e^{i\theta}$$

$$\theta = \omega t \rightarrow e^{i\omega t}, e^{-i\omega t}$$

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

4. DFT

$$\hat{f}_k = \sum_{j=0}^{n-1} f_j e^{\frac{-i2\pi jk}{n}} \qquad \omega_n = e^{\frac{-2\pi i}{n}}$$

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \dots \\ \hat{f}_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \dots \\ f_{n-1} \end{bmatrix}$$

5. FFT

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \dots \\ \hat{f}_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \dots \\ f_{n-1} \end{bmatrix}$$

$$\hat{f}_{n-1} = f_0 + f_1\omega_n^{n-1} + f_2\omega_n^{2(n-1)} + \dots + f_{n-1}\omega_n^{(n-1)(n-1)}$$

$$A(x) = \sum_{j=0}^{n-1} a_j x^j$$

$$= a_0x^0 + a_1x^1 + a_2x^2 + \dots + a_{n-1}x^{n-1}$$

1. Divide

$$A_{even}(x) = \sum_{k=0} a_{2k}x^k =< a_0, a_2, a_4, \ldots >$$

$$A_{add}(x) = \sum_{k=0} a_{2k+1}x^k =< a_1, a_3, a_5, \ldots >$$

1. Conquer: recursively compute

$$A_{even}(y) \ \& \ A_{add}(y) \ \ for \ y \in x^2 = \{x^2 | x \in X\}$$

1. Combine

$$A(x) = A_{even}(x^2) + x * A_{add}(x^2)$$

$$for \ x \in X$$

$$T(n,|x|) = 2T(n/2,|x|) + O(n+|x|)$$

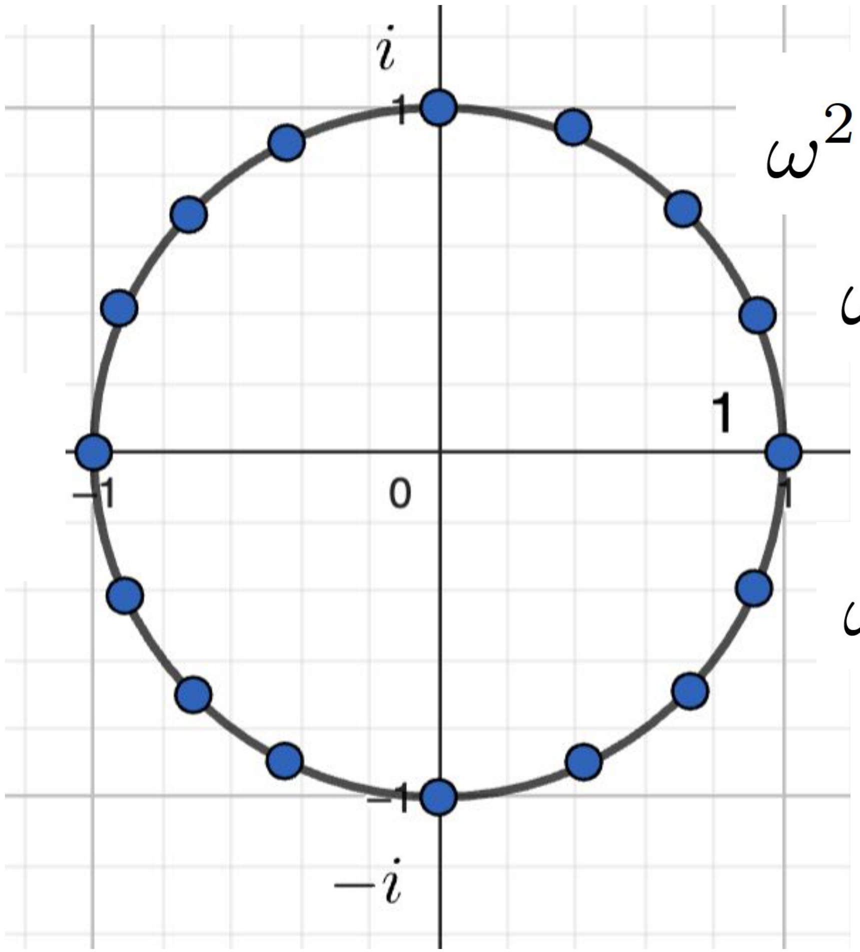
$$T(n) = 2T(n/2) + O(n)$$

N^{th} Roots of Unity

$$e^{i\theta} = cos(\theta) + isin(\theta)$$

$$Z^n = 1$$

$$\omega = e^{\frac{2\pi i}{n}}$$



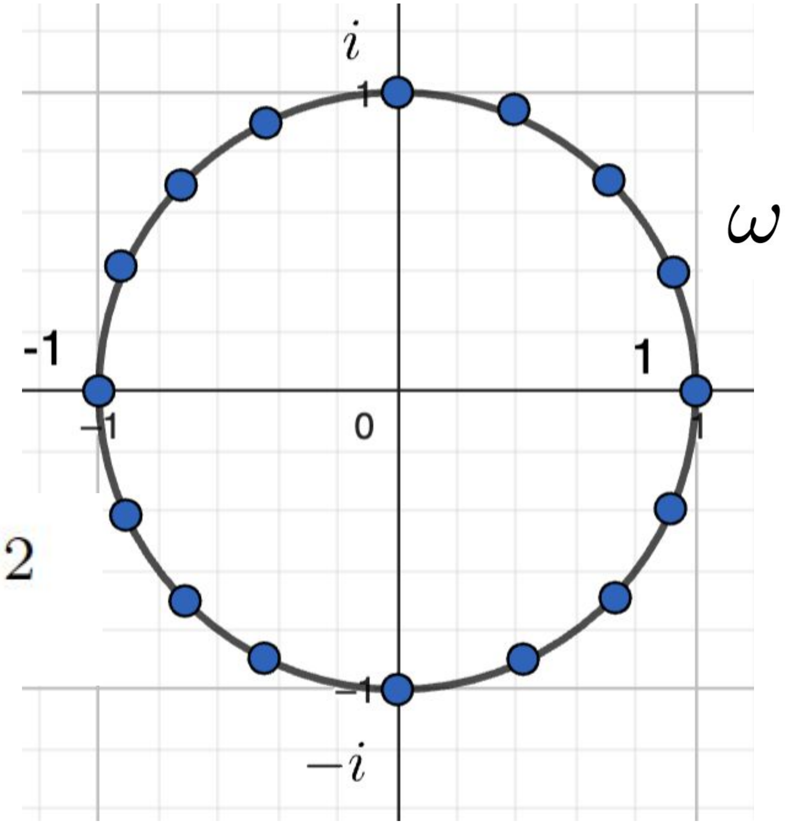
$$\omega^2 = e^{\frac{2\pi(2)i}{n}}$$

$$\omega^1 = e^{\frac{2\pi i}{n}}$$

$$\omega^0 = 1$$

$$\omega^{n-1} = e^{\frac{2\pi(n-1)i}{n}}$$

$$\omega^{j+n/2} = -\omega^{j} \rightarrow (\omega^{j}, \omega^{j+n/2}) \ are \pm paired$$



$$\omega^{1} = e^{\frac{2\pi i}{n}}$$

$$\omega^{1+n/2}$$

$Evaluate\ A_{even}(x^2) and\ A_{odd}(x^2)\ at\ [1, \omega^2, \omega^4, ..., \omega^{2(n-1)}]$

$Evaluate\ A(x)\ at\ [1, \omega, \omega^2, ..., \omega^{n-1}]$

$(n/2)\ roots\ of\ unity$

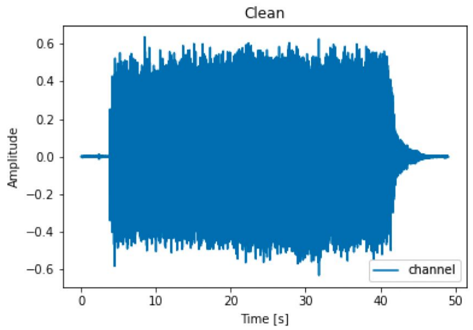
$$\hat{f}_k = \sum_{j=0}^{n-1} f_j e^{\frac{-i2\pi jk}{n}} \qquad \omega_n = e^{\frac{-2\pi i}{n}}$$

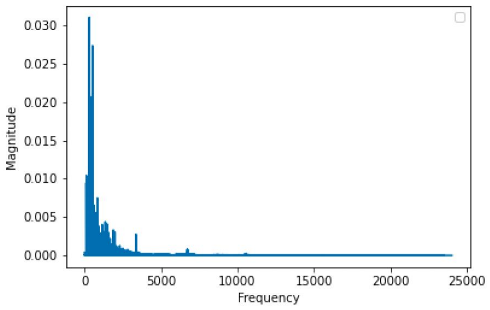
$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \dots \\ \hat{f}_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \dots \\ f_{n-1} \end{bmatrix}$$

$$A(x) = \sum_{j=0}^{n-1} a_j x^j$$

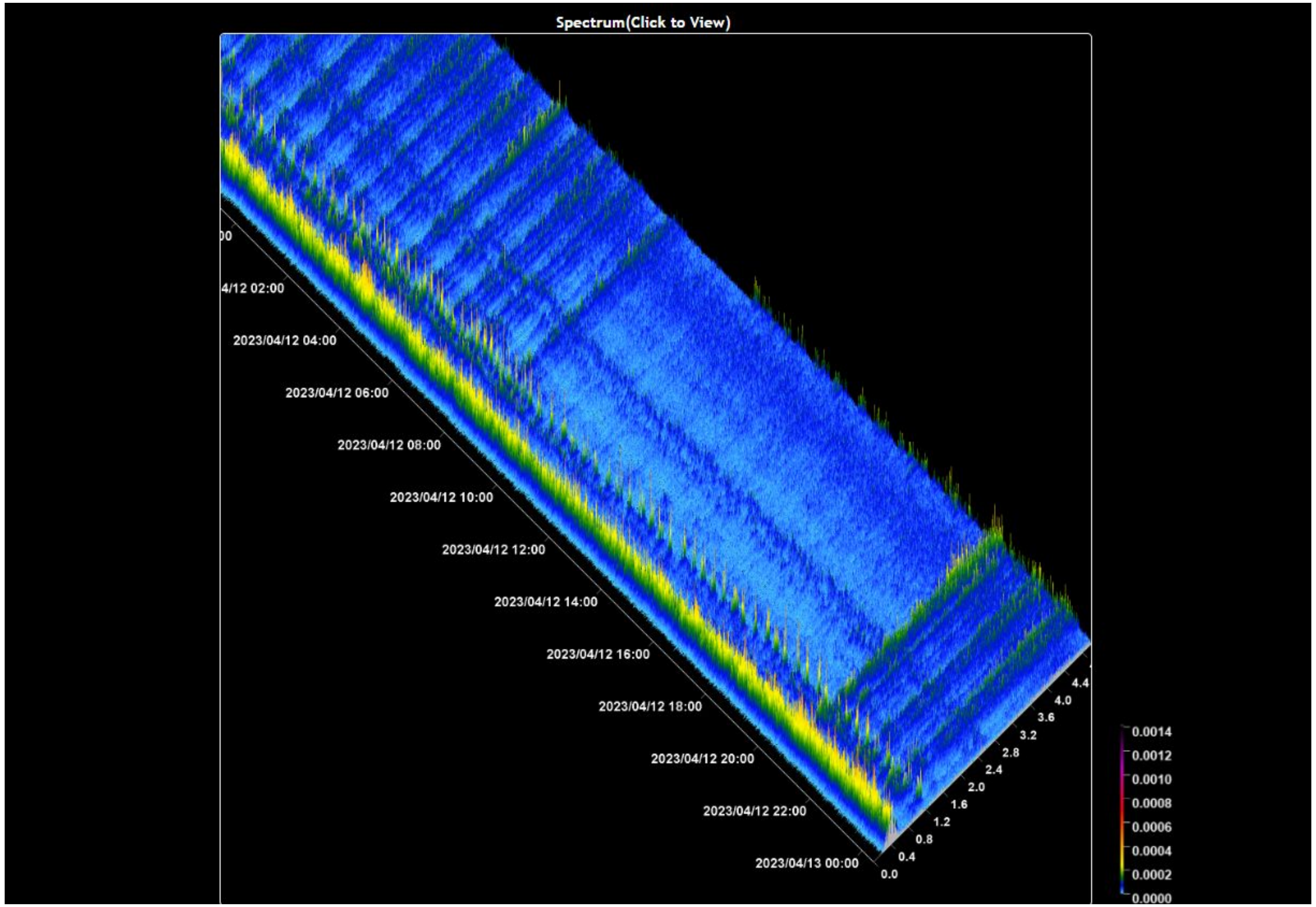
$$= a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots + a_{n-1} x^{n-1}$$

6. Application

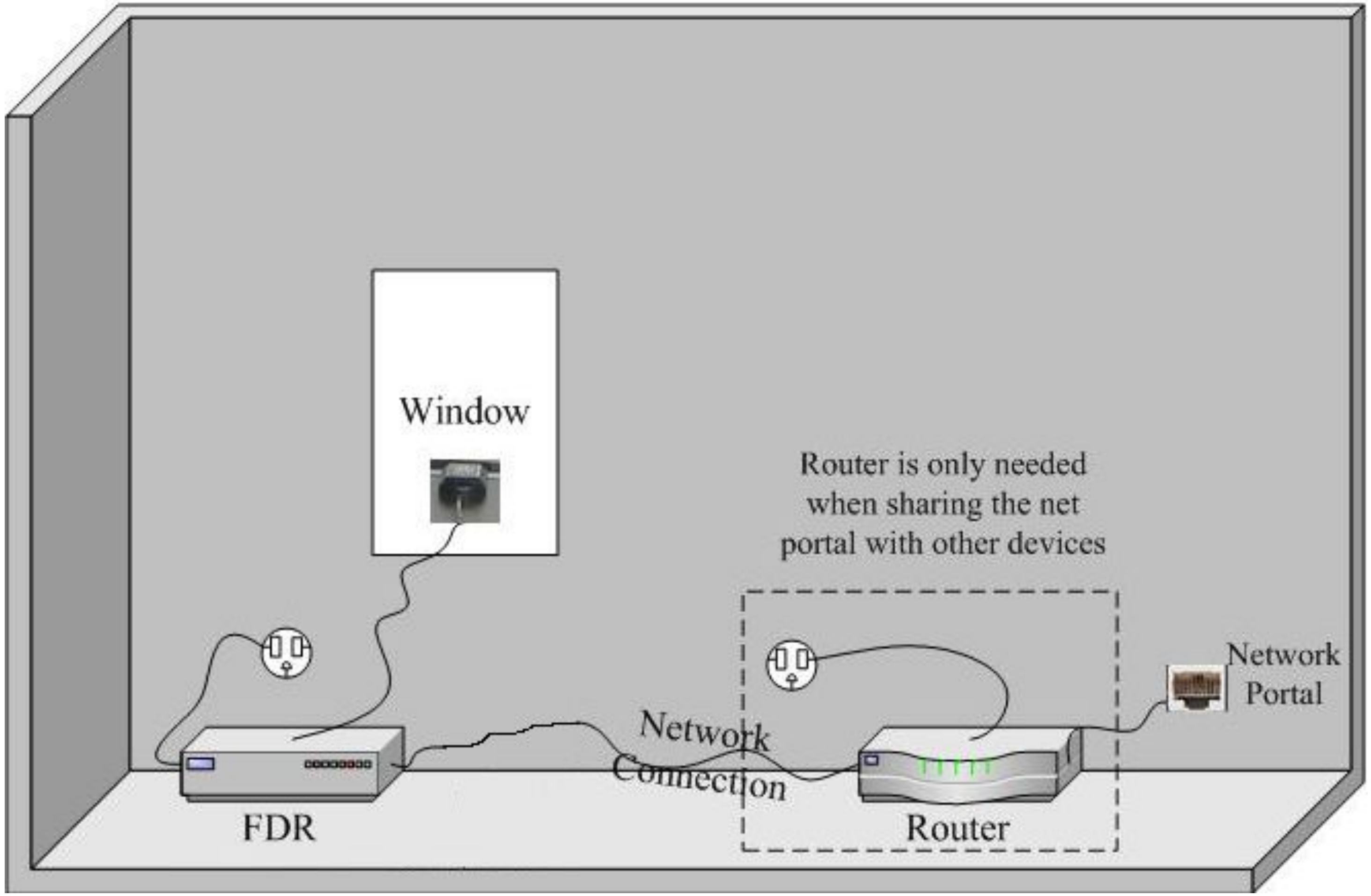




7. Implementation



Frequency Disturbance Recorder(FDR)



Angle	1.3182
ConvNum	7
CurrentAng	0
CurrentMag	0
Date	41323
FDRID	730
FDRName	"UsMTMduglendive730"
FPS	10
Frequency	60.0102
HasAngle	true
HasCurrentAng	false
HasCurrentMag	false
HasFreqAndAng	true
HasFrequency	true
HasVoltage	true
POWIList	null
POWVList	null
RelativeAngle	0
Time	140405
TimeStamp	{04/13/2023 14:04:05}
TimestampFractionalSec	0.6
TimestampToSecond	{04/13/2023 14:04:05}
UserObj	null
Voltage	121.859
_angle	1.3182
_currentAng	0
_currentMag	0
_frequency	60.0102
_relativeAngle	0
_valueStatus	7

Count = 1200
60.0094986
60.00979996
60.00999832
60.01029968
60.00910187
60.00870132
60.0094986
60.00889969
60.0082016
60.00830078
60.00870132
60.00749969
60.00849915
60.00699997
60.00759888
60.00709915

FFT

{(0, 7.02448096024606E-07)}
{(0.00834028356964137, 7.1002095795584E-07)}
{(0.0166805671392827, 7.32353557209559E-07)}
{(0.0250208507089241, 7.68406678299135E-07)}
{(0.0333611342785655, 8.16786567258898E-07)}
{(0.0417014178482068, 8.76007525368573E-07)}
{(0.0500417014178482, 9.44616631506574E-07)}
{(0.0583819849874896, 1.02685546251439E-06)}
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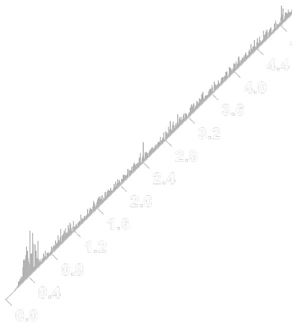
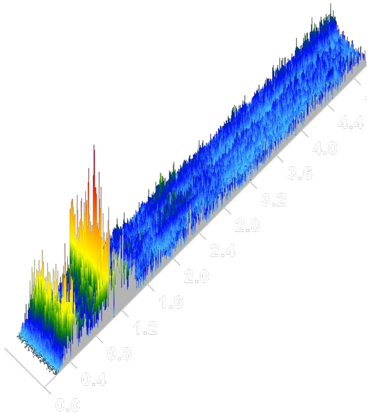
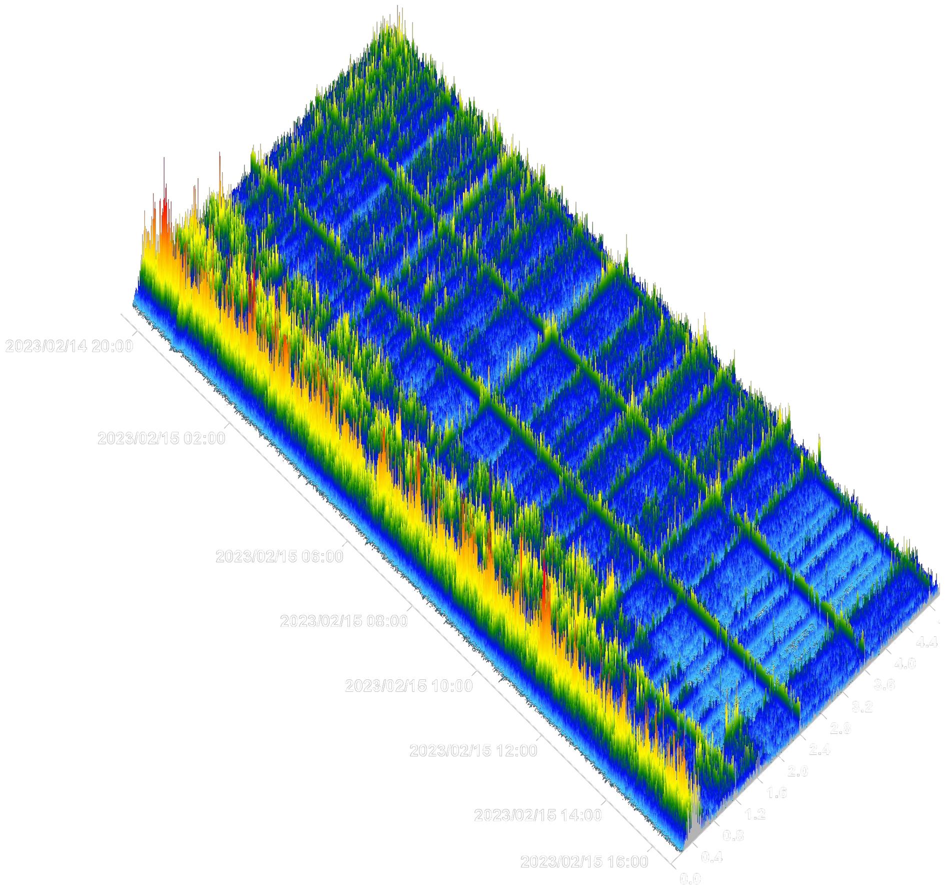






https://fnetpublic.utk.edu/spectrum_map.html

8. References

[1] https://en.wikipedia.org/wiki/Fourier_series

[2] https://en.wikipedia.org/wiki/Fourier_transform

[3] https://en.wikipedia.org/wiki/Discrete_Fourier_transfo

[4] https://en.wikipedia.org/wiki/Fast_Fourier_transform

[5] https://eipi10.cn/mathematics/2020/04/19/fourier_transform_2/

[6]https://blog.csdn.net/qq_29545231/article/details/108547437

Questions

1. Does the Fourier series prove that a non-periodic function is a sum of trigonometric functions?

2. What are the steps of divide & conquer approach?

3. Based on Euler's formula, a general point on a complex plane can be represented in what format?

Thanks

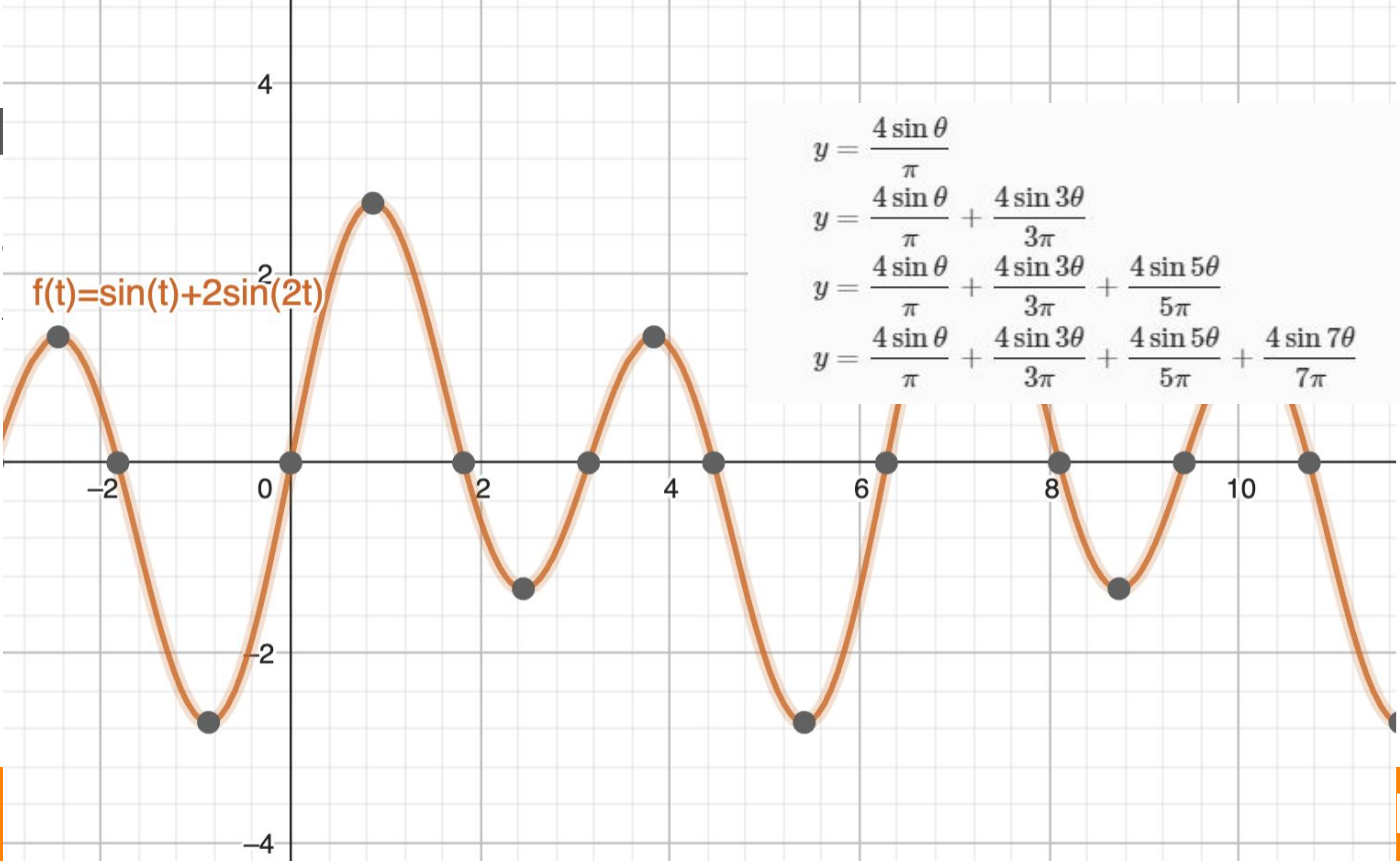
4. DFT

$$A(x) = \sum_{j=0}^{n-1} a_j x^j$$

$$= a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots + a_{n-1} x^{n-1}$$

$$\hat{f}_{n-1} = f_0 + f_1 \omega_n^{n-1} + f_2 \omega_n^{2(n-1)} + \ldots + f_{n-1} \omega_n^{(n-1)(n-1)}$$

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \ldots \\ \hat{f}_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n & \omega_n^2 & \ldots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \ldots & \omega_n^{2(n-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \ldots \\ f_{n-1} \end{bmatrix}$$

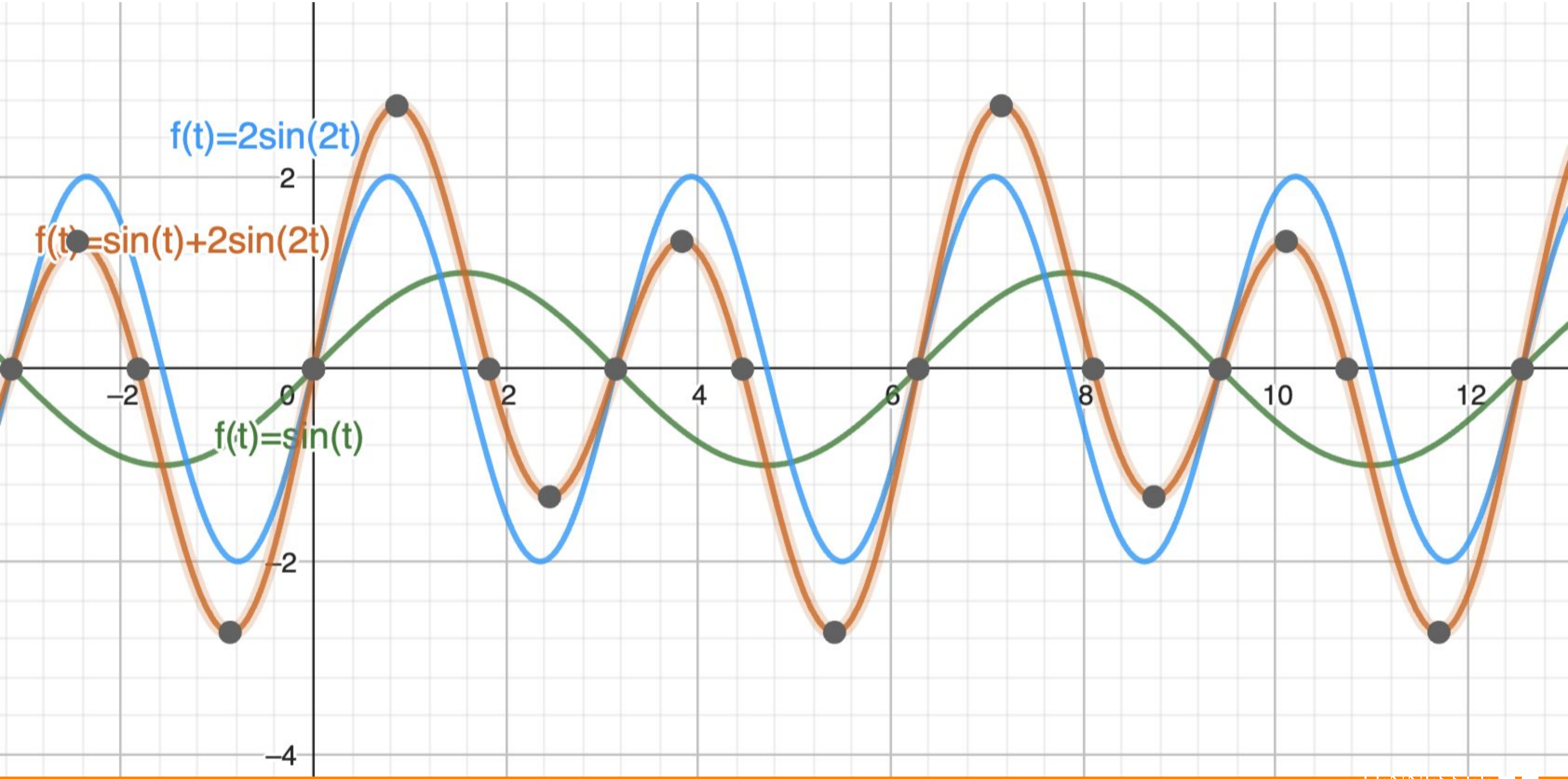


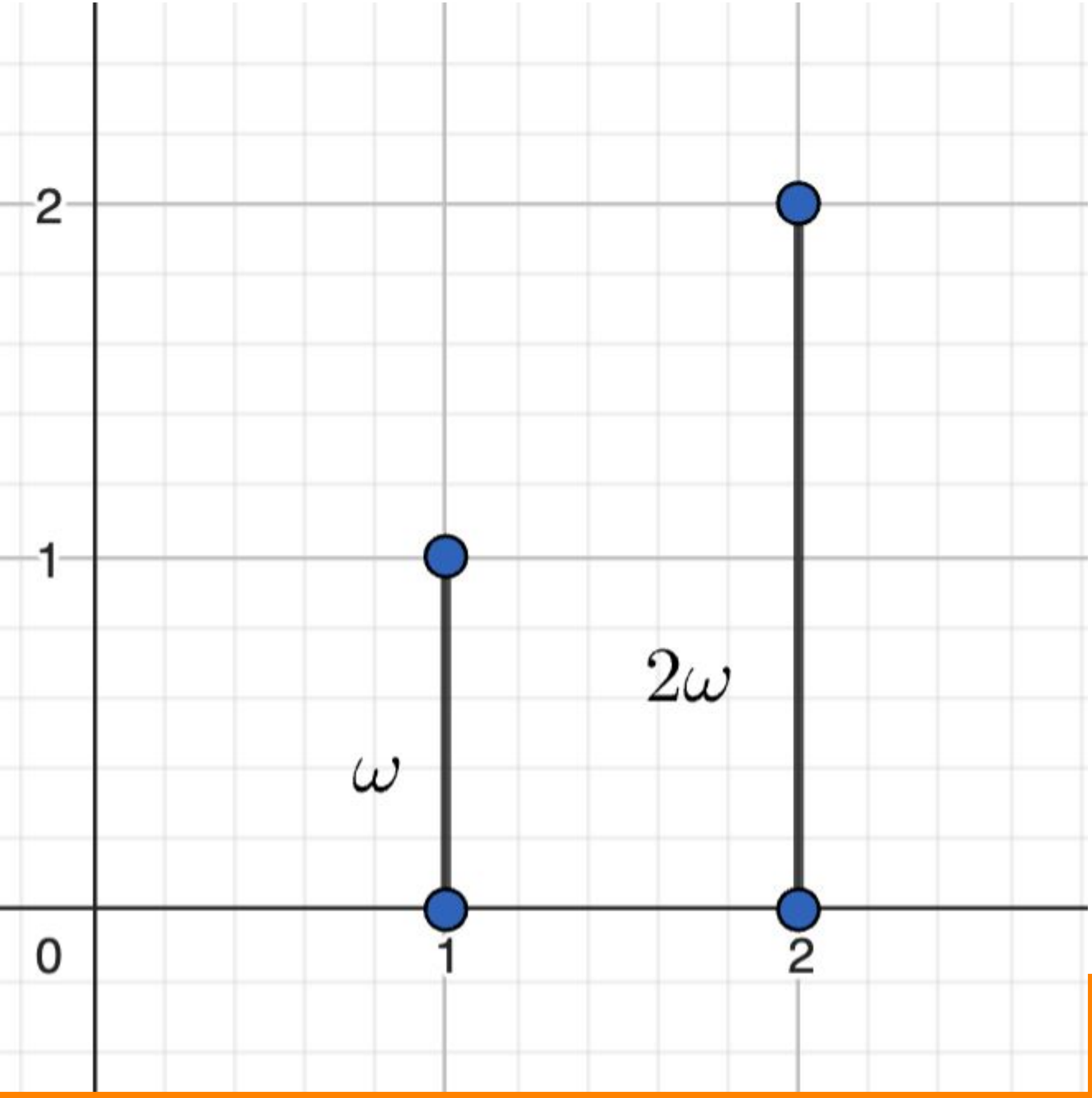
$$y = \frac{4\sin\theta}{\pi}$$

$$y = \frac{4\sin\theta}{\pi} + \frac{4\sin 3\theta}{3\pi}$$

$$y = \frac{4\sin\theta}{\pi} + \frac{4\sin 3\theta}{3\pi} + \frac{4\sin 5\theta}{5\pi}$$

$$y = \frac{4\sin\theta}{\pi} + \frac{4\sin 3\theta}{3\pi} + \frac{4\sin 5\theta}{5\pi} + \frac{4\sin 7\theta}{7\pi}$$


f(t)=2sin(2t)
f(t)=sin(t)+2sin(2t)
f(t)=sin(t)
2
−2
−4
−2
0
2
4
6
8
10
12



$$\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \begin{cases} = 0 \\ \neq 0 \end{cases}$$

5. FFT

First, we are going to divide f(x) into even & odd coefficients

Next, recursively compute A even and A odd for y in x^2 where x2 is the set of squares of all numbers in x.

Combine

Questions

1. Does the Fourier series prove that a non-periodic function is a sum of trigonometric functions?

2. What are the steps of divide & conquer approach?

3. Based on Euler's formula, a general point on a complex plane can be represented in what format?

Background

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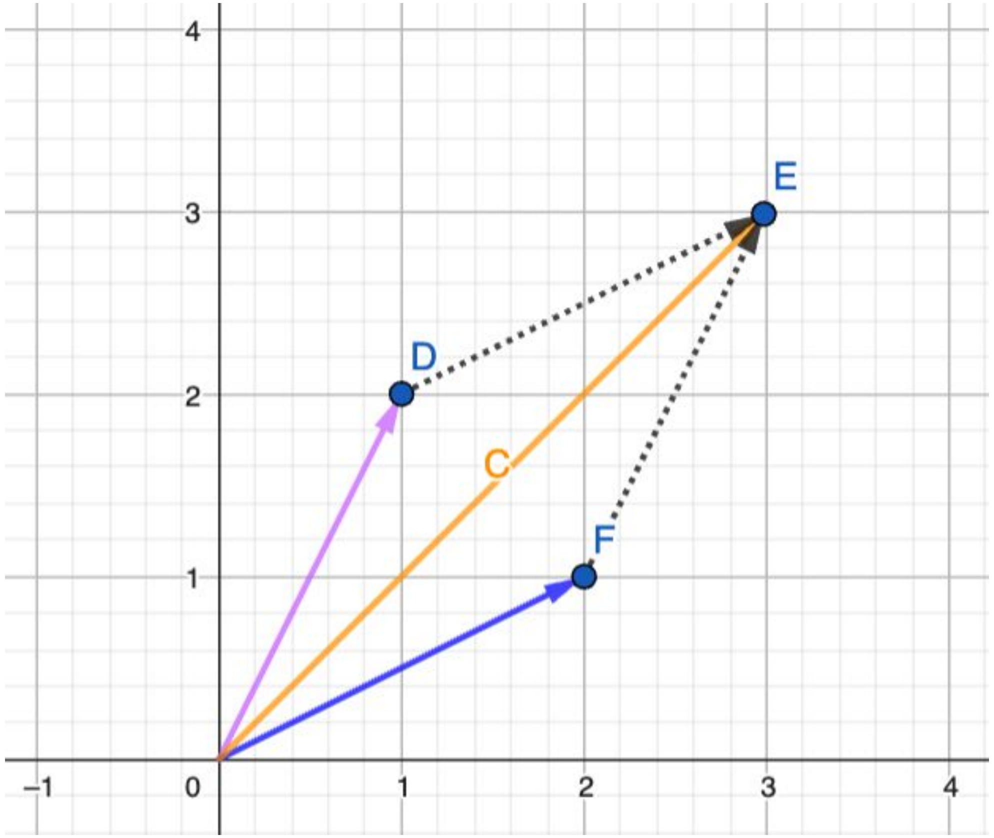

Background

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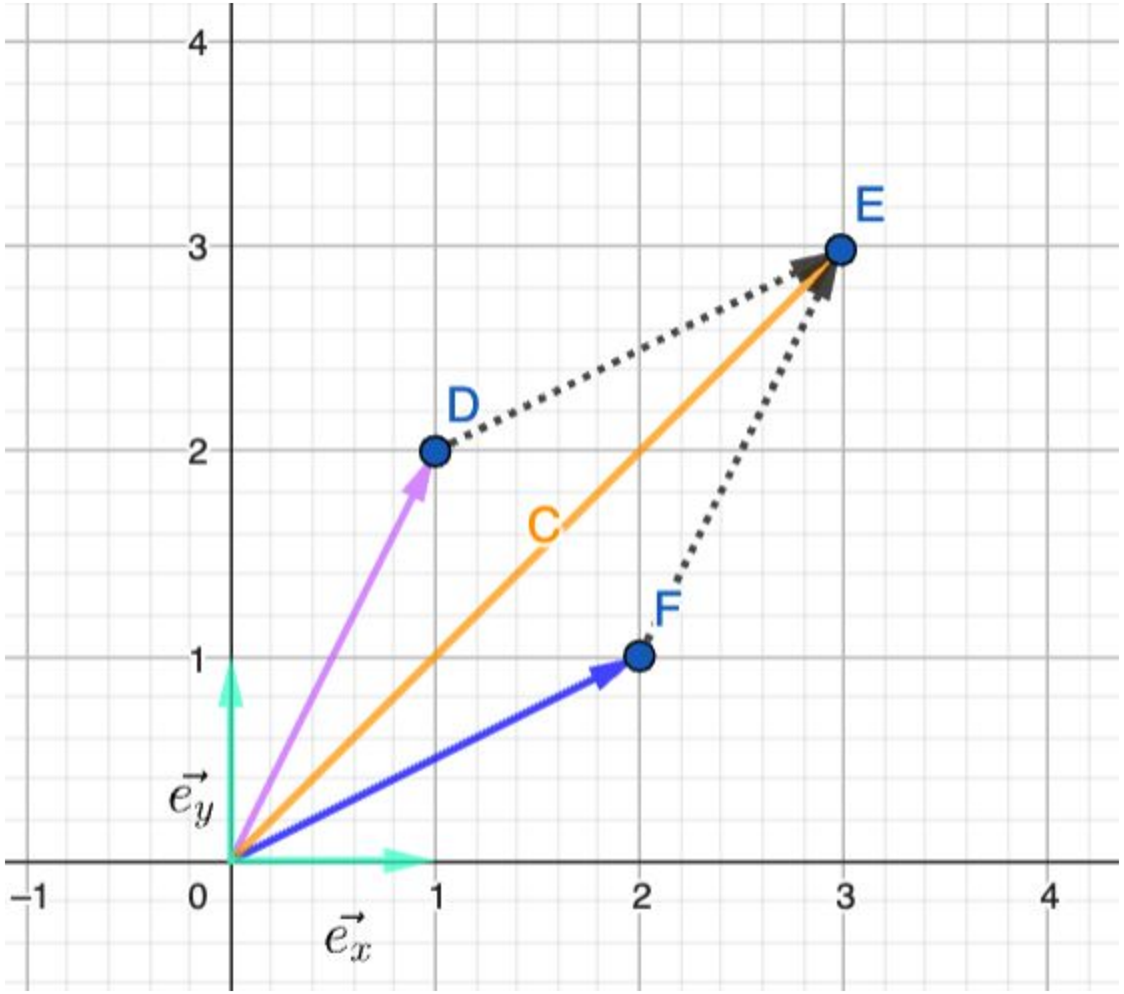
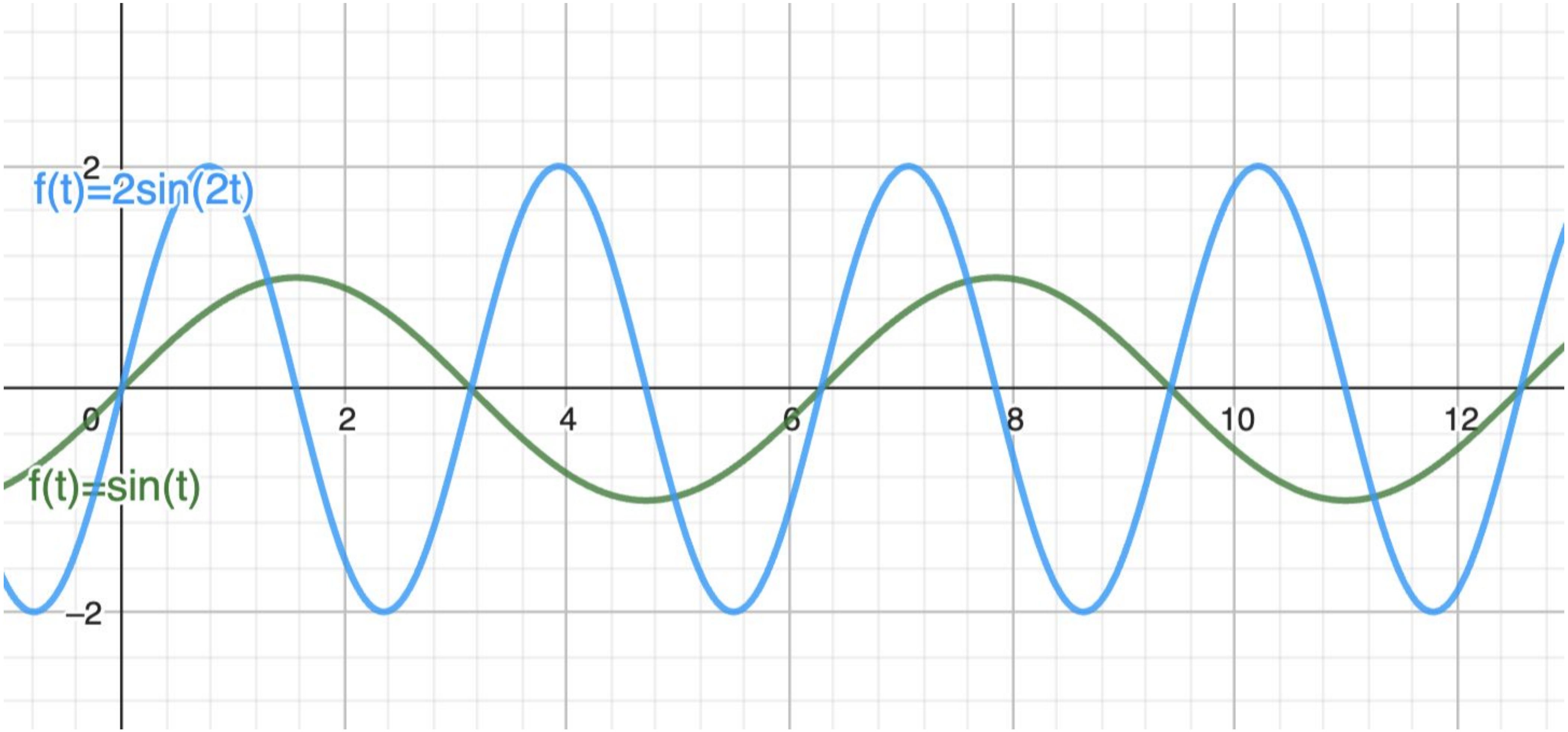
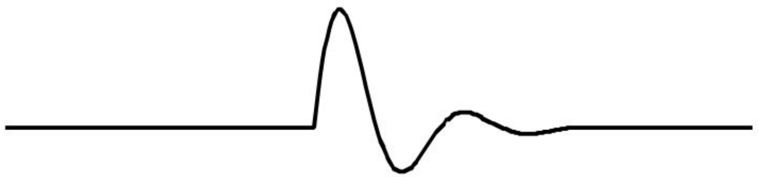
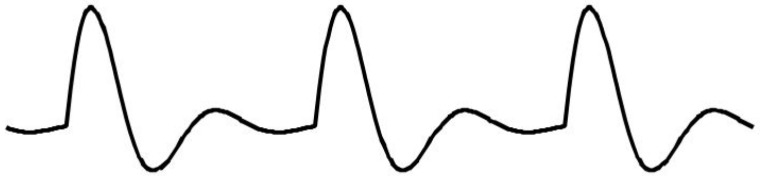



f(t)=2sin(2t)
f(t)=sin(t)
2
–2
0
2
4
6
8
10
12

Type of Transform	Example Signal
Fourier Transform *signals that are continious and aperiodic*	
Fourier Series *signals that are continious and periodic*	
Discrete Time Fourier Transform *signals that are discrete and aperiodic*	
Discrete Fourier Transform *signals that are discrete and periodic*	

[5] **The Scientist and Engineer's Guide to Digital Signal Processing.** Steven W. Smith California Technical Pub., 1997 - Digital filters (Mathematics). Chap 8, Page 145.