CS581 Master Theorem Examples

Recall:

Master Theorem:
Let $a \geq 1$ and $b \geq 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined by the recurrence:

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to be either ceil(n/b) or floor(n/b). Then $T(n)$ has the following asymptotic bounds:

Case 1. If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then

$$T(n) = \Theta(n^{\log_b a})$$

Case 2. If $f(n) = \Theta(n^{\log_b a})$, then

$$T(n) = \Theta(n^{\log_b a} \lg n)$$

Case 3. If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, AND if $a f(n/b) \leq c f(n)$ for some constant $c < 1$ as $n \to +\infty$, then, $T(n) = \Theta(f(n))$

Note:
A little trick –
$f(n)$ often takes the form:

$$f(n) = n^k \log^p n,$$

where $k \geq 0$ and $p \in \mathbb{R}$.

If $a > b^k \Rightarrow$ case 1
If $a = b^k \Rightarrow$ case 2
If $a < b^k \Rightarrow$ case 3

Ex 1 $T(n) = 16T\left(\frac{n}{4}\right) + n^2$

$a = 16,\ b = 4,\ f(n) = n^2$

$\Rightarrow n^{\log_b a} = n^{\log_4 16} = n^2$

<u>case 2</u>: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$

proof: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2}{n^2} = 1 \neq 0, \infty \Rightarrow f = \Theta(g)$.

So, $T(n) = \Theta(n^2 \lg n)$ $\square$

Ex2 $T(n) = 2T\left(\frac{n}{4}\right) + 1$

$a = 2,\ b = 4,\ f(n) = 1$

$\Rightarrow n^{\log_b a} = n^{\log_4 2} = n^{\frac{1}{2}}$

<u>Case 1</u>: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

proof: Let $\varepsilon = \frac{1}{2} > 0$, then
$f(n) = O(n^{\frac{1}{2} - \frac{1}{2}}) = O(1)$

$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} 1 = 1 \neq \infty \Rightarrow f = O(g)$.

So, $T(n) = \Theta(n^{\frac{1}{2}})$

$\square$

Ex 3 $T(n) = 3T(\frac{n}{2}) + n^2 \log^3 n$

$a = 3,\ b = 2,\ f(n) = n^2 \log^3 n$

$\Rightarrow n^{\log_b a} = n^{\log_2 3}$.

Case 3: If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(f(n))$.

proof: Let $\varepsilon = \log_2(8/3) > 0$, then

$$f(n) = \Omega(n^{\log_2 3 + \log_2(8/3)}) = \Omega(n^2)$$

$$\Rightarrow \lim_{n\to\infty} \frac{n^2 \log^3 n}{n^2} = \lim_{n\to\infty} \log^3 n = \infty \neq 0 \Rightarrow f = \Omega(g)$$

so, $T(n) = \Theta(n^2 \log^3 n)$

$\square$

check regularity condition:
$a f(n/b) \leq c f(n)$ for some $c < 1$
as $n \to +\infty$.

proof: Let $c = \frac{1}{2} < 1$.

$$\Rightarrow 3\left(\left(\frac{n}{2}\right)^2 \log^3\left(\frac{n}{2}\right)\right) \leq \frac{1}{2}\left(n^2 \log^3 n\right)$$
as $n \to +\infty$.

$\square$

Note: The master theorem does not always apply. $f(n)$ must be polynomially larger/smaller than $n^{\log_b a}$.

$\Rightarrow$ see textbook pg. 95.

EX 4 $T(n) = 4T\left(\frac{n}{4}\right) + \frac{n}{\log n}$

$a=4,\ b=4,\ f(n) = \frac{n}{\log n}$

$\Rightarrow n^{\log_b a} = n^{\log_4 4} = n^1 = n$

Case 3: $\frac{f(n)}{\log_b a} = \frac{n}{n \log n} = \frac{1}{\log n} < n^{1+\varepsilon} \ \forall \varepsilon > 0$

ratio between $f(n)$ and $n^{\log_b a}$ is non-polynomial.

Master theorem does not apply. $\square$

* see textbook pgs 95-96 for more examples.