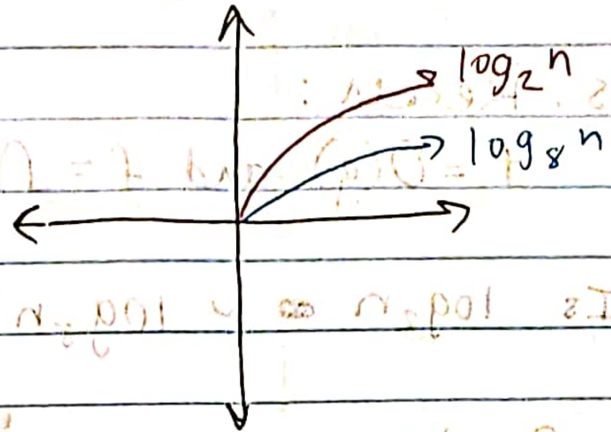
CS581 Asymptotics Examples

EX 1 $f(n) = \log_2 n$ $g(n) = \log_8 n$



Q $\boxed{1}$ Is $\log_2 n = \Omega(\log_8 n)$?

$\Rightarrow$ Does $c > 0$ and n_0 exist $\ni$

$c(\log_8 n) \leq \log_2 n \quad \forall n \geq n_0$?

A Yes. By Picture $\Rightarrow$ $c = 1$, $n_0 = 0$ satisfies definition. $\square$

Q $\boxed{2}$ Is $\log_2 n = O(\log_8 n)$?

$\Rightarrow$ Does $c > 0$ and n_0 exist $\ni$

$0 \leq \log_2 n \leq c(\log_8 n) \quad \forall n \geq n_0$?

A Yes. Proof:

$$\log_8 n = \frac{\log_2 n}{\log_2 8} = \frac{1}{\log_2 8} \log_2 n \quad \text{(by change of base)}$$

Thus, $0 \leq \log_2 n \leq c \cdot \frac{1}{\log_2 8} \log_2 n$

so, $c = 3$, $n_0 = 1$ by def. $\square$

Q $\boxed{3}$ Is $\log_2 n = \Theta(\log_8 n)$?

A Yes. Recall:

$$f = O(g) \text{ and } f = \Omega(g) \Leftrightarrow f = \Theta(g).$$

$\square$

Q $\boxed{4}$ Is $\log_2 n \sim \log_8 n$?

A NO. Proof:

If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \Rightarrow f \sim g;$

so, $\lim_{n\to\infty} \frac{\log_2 n}{\log_8 n}$

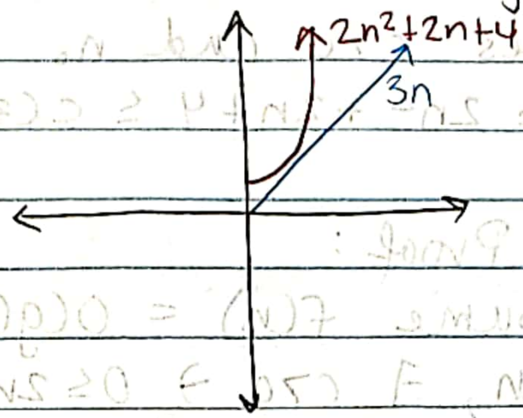
$= \lim_{n\to\infty} \frac{\log_2 n}{\frac{1}{\log_2 8}\log_2 n}$

$= \log_2 8 \lim_{n\to\infty} \frac{\log_2 n}{\log_2 n}$

$= 3 \lim_{n\to\infty} 1 = 3 \neq 1.$

$\square$

Ex 2) $f(n) = 2n^2+2n+4 \qquad g(n) = 3n$



Q $\boxed{1}$ Is $2n^2+2n+4 = \Omega(3n)$?

$\Rightarrow$ Does $c > 0$ and n_0 exist $\ni$

$c(3n) \le 2n^2+2n+4 \quad \forall n \ge n_0$?

A Yes. Proof:

If $n \ge 1$, then $n \le n$ and $n \le n^2$.

Therefore,

$$\Rightarrow 2n \le 2n^2$$

$$\Rightarrow 2n+2n \le 2n^2+2n$$

$$\Rightarrow 4n+4 \le 2n^2+2n+4$$

$$\Rightarrow 4n \le 4n+4 \le 2n^2+2n+4$$

$$\Rightarrow 4n \le 2n^2+2n+4.$$

So, $c = \frac{4}{3}$, $n_0 = 1 \Rightarrow f(n) = \Omega(g(n))$ by definition.

$\square$

Q [2] . Is $2n^2 + 2n + 4 = O(3n)$?

$\Rightarrow$ Does $c > 0$ and n_0 exist $\ni$

$0 \le 2n^2 + 2n + 4 \le c(3n) \ \forall n \ge n_0$?

A No. Proof:

Assume $f(n) = O(g(n))$.

Then, $\exists\ c > 0 \ni 0 \le 2n^2+2n+4 \le c(3n)$

as $n \to +\infty$ by definition.

so, $\dfrac{0 \le 2n^2+2n+4 \le c(3n)}{2n^2+2n+4}$ (divide by $f(n)$)

$$\Rightarrow 0 \le 1 \le \frac{c(3n)}{2n^2+2n+4} \quad \left(\text{take } \lim_{n\to\infty}\right)$$

$$\Rightarrow 0 \le 1 \le \lim_{n\to\infty} \frac{c(3n)}{2n^2+2n+4}$$

$$3 \cdot c \lim_{n\to\infty} \frac{n}{2n^2+2n+4}$$

$$3 \cdot c \lim_{n\to\infty} \frac{\frac{1}{n}}{2+\frac{2}{n}+\frac{4}{n^2}}$$

$$3 \cdot c \cdot \left(\frac{0}{2}\right) = 0$$

$$\Rightarrow 0 \le 1 \le 0.$$

contradiction, therefore

$2n^2+2n+4 \ne O(3n)$.

$\square$

Q [3] Is $f(n) = \omega(g(n))$?

A Yes. Proof:

If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \Rightarrow f(n) = \omega(g(n))$.

So, $\lim_{n \to \infty} \frac{2n^2 + 2n + 4}{3n} = \frac{\infty}{\infty}$

Apply ~~*~~ L'Hospitals Rule $\left[\lim_{n \to \infty} \frac{4n + 2}{3} = \infty \right.$

$\square$

A Alternate proof:

Recall: $\lim_{n \to \infty} \frac{3n}{2n^2 + 2n + 4} = \frac{g(n)}{f(n)} = 0$

and,

If $\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0 \Rightarrow g = o(f)$

and,

If $g = o(f)$, then $f = \omega(g)$.

$\square$